

ON THE ZERO FORCING NUMBER OF CORONA AND LEXICOGRAPHIC PRODUCT OF GRAPHS

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ABSTRACT. The zero forcing number of a graph G , denoted by $Z(G)$, is the minimum cardinality of a set S of black vertices (where vertices in $V(G) \setminus S$ are colored white) such that $V(G)$ is turned black after finitely many applications of “the color change rule”: a white vertex is turned black if it is the only white neighbor of a black vertex. In this paper, we study the zero forcing number of corona product, $G \odot H$ and lexicographic product, $G \circ H$ of two graphs G and H . It is shown that if G and H are connected graphs of order $n_1 \geq 2$ and $n_2 \geq 2$ respectively, then $Z(G \odot^k H) = Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1} Z(H)$, where $G \odot^k H = (G \odot^{k-1} H) \odot H$. Also, it is shown that for a connected graph G of order $n \geq 2$ and an arbitrary graph H containing $l \geq 1$ components H_1, H_2, \dots, H_l with $|V(H_i)| = m_i \geq 2$, $1 \leq i \leq l$, $(n-1)l + \sum_{i=1}^l m_i \leq Z(G \circ H) \leq n(\sum_{i=1}^l m_i) - l$.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a simple, undirected, connected graph with $|V(G)| \geq 2$. The number of vertices and edges of G are called the order and the size of G respectively. The degree of a vertex $v \in V$, denoted by $\deg_G(v)$, is the number of edges incident to the vertex v in G . If there is no ambiguity, we will use the notation $\deg(v)$ instead of $\deg_G(v)$. An end vertex is a vertex of degree one. Given $u, v \in V$, $u \sim v$ means that u and v are adjacent vertices and $u \not\sim v$ means that u and v are not adjacent. We define the open neighborhood of a vertex v in G , $N_G(v) = \{u \in V(G) : u \sim v\}$ and the closed neighborhood of v , $N_G[v] = N_G(v) \cup \{v\}$. If there is no ambiguity, we will simply write $N(v)$ or $N[v]$. If $u \in N_G(v)$ then u is said to be a neighbor of v . We denote a path, cycle, complete graph and empty graph on n vertices by P_n , C_n , K_n and $\overline{K_n}$ respectively. All graphs considered in this paper are non

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trivial unless otherwise stated.

The notion of a zero forcing set, as well as the associated zero forcing number, of a simple graph was introduced in [1] to bound the minimum rank of associated matrices for numerous families of graphs. Let each vertex of a graph G be given one of two colors, “black” and “white” by convention. Let S denote the initial set of black vertices of G . The color-change rule converts the color of a vertex u_2 from white to black if the white vertex u_2 is the only white neighbor of a black vertex u_1 ; we say that u_1 forces u_2 , which we denote by $u_1 \rightarrow u_2$. And a sequence, $u_1 \rightarrow u_2 \rightarrow \cdots \rightarrow u_i \rightarrow u_{i+1} \rightarrow \cdots \rightarrow u_t$, obtained through iterative applications of the color-change rule is called a forcing chain. The set S is said to be a zero forcing set of G if all the vertices of G will be turned black after finitely many applications of the color-change rule. The zero forcing number of G , denoted by $Z(G)$, is the minimum of $|S|$ over all zero forcing sets $S \subseteq V(G)$. A zero forcing set of cardinality $Z(G)$ is called a forcing basis for G . For surveys on the zero forcing parameter, see [9, 10]. For more on the zero forcing parameter in graphs, see [2, 3, 5, 6, 7, 12].

If F is a field, $M_n(F)$ denotes the set of all $n \times n$ matrices over F . An n -square matrix A is said to be a symmetric matrix if $A^T = A$. The set of all real symmetric n -square matrices is denoted by S_n . To a given graph G with vertex set $\{1, 2, \dots, n\}$, we associate a class of real, symmetric matrices as follows:

$$S(G) = \{A = [a_{ij}] | A \in S_n, \text{ for } i \neq j, a_{ij} \neq 0 \Leftrightarrow ij \in E(G)\}.$$

Note that there is no restriction on the value of a_{ii} with $i = 1, 2, \dots, n$ and the adjacency matrix $A(G)$ belongs to $S(G)$, where the adjacency matrix of a graph G is a square $(0, 1)$ -matrix of size n , whose (i, j) -th entry is 1 if and only if v_i is adjacent to v_j , since there are no loops in the graph, the diagonal entries of the adjacency matrix are zero. On the other hand, the graph of an n -square symmetric matrix A , denoted by $\mathcal{G}(A)$, is the graph with vertices $\{1, 2, \dots, n\}$ and the edge set

$$\{ij | a_{ij} \neq 0, 1 \leq i \neq j \leq n\}.$$

The minimum rank of G is defined to be

$$mr(G) = \min\{rank(A) | A \in S(G)\},$$

while the maximum nullity of G is defined as

$$M(G) = \max\{null(A) | A \in S(G)\}.$$

We have

$$mr(G) + M(G) = |V(G)|.$$

The underlying idea for the zero forcing set of a graph is that a black vertex is associated with a coordinate in a vector that is required to be zero, while a white vertex indicates a coordinate that can be either zero or nonzero. Changing a vertex from white to black is essentially noting that the corresponding coordinate is forced to be zero if the vector is in the kernel of a matrix in $S(G)$ and all black vertices indicate coordinates assumed to be or previously forced to be 0. Hence the use of the term “zero forcing set”, see [1].

The support of a vector $x = (x_i)$, denoted by $Supp(x)$, is the set of indices i such that $x_i \neq 0$. Let Z be a zero forcing set of G and $A \in S(G)$. If $x \in null(A)$ and $Supp(x) \cap Z = \phi$, then $x = 0$, stated in [1, 14]. Also from [1, 14], we have $M(G) \leq Z(G)$ for a graph G .

In this paper, we consider corona product and lexicographic product of graphs in the context of zero forcing number. This paper consists of three sections. Section 1 includes introduction. Sections 2 and 3 include several results related to the zero forcing number of corona and lexicographic product of graphs, respectively.

2. CORONA PRODUCT OF GRAPHS

Let G and H be two graphs of order n_1 and n_2 respectively. The corona product of G and H is defined as the graph obtained from G and H by taking one copy of G and n_1 copies of H and joining by an edge each vertex from the i^{th} -copy of H with the i^{th} -vertex of G . We will denote by $V = \{v_1, v_2, \dots, v_{n_1}\}$, the set of vertices of G and by $H_i = (V_i, E_i)$, the i -th copy of H , where $V_i = \{u_1^i, u_2^i, \dots, u_{n_2}^i\}$, such that $v_i \sim u_k^i$ for every $u_k^i \in V_i$. Note that the subgraph of $G \odot H$ induced by V_i is H_i and the corona graph $K_1 \odot H$ is isomorphic to the join graph $K_1 + H$. For any integer $k \geq 2$, we define the graph $G \odot^k H$ recursively from $G \odot H$ as $G \odot^k H = (G \odot^{k-1} H) \odot H$. It is also noted that $|G \odot^{k-1} H| = n_1(n_2 + 1)^{k-1}$ and $|G \odot^k H| = |G \odot^{k-1} H| + n_1 n_2 (n_2 + 1)^{k-1}$.

We call the copies of H in $G \odot H$ as the copies of H in 1st-corona, the newly added copies of H in $G \odot H$ to obtain $G \odot^2 H$ as the copies of H in 2nd-corona and generally the newly added copies of H in $G \odot^{k-1} H$ to obtain $G \odot^k H$ as the copies of H in k^{th} -corona.

In $G \odot^k H$ for any positive integer k , we name the vertices in $G \odot^{k-1} H$ as the root vertices of the copies of H in k^{th} -corona, that are joined to these vertices in $G \odot^k H$.

As one can color the vertices of $G \odot^k H$ in more than one ways, but in this paper for a disconnected graph H (containing isolated vertices) of order at least two, we will consider the zero forcing set of $G \odot^k H$, that contains only the vertices of H but not the vertices of G .

In Figure 1, the graph with grey vertices is $G = P_2$, the copies of H with black vertices are the copies of H in first corona and with white vertices are the copies of H in 2nd corona. The black and grey vertices are the root vertices of the corresponding copies of H with white vertices.

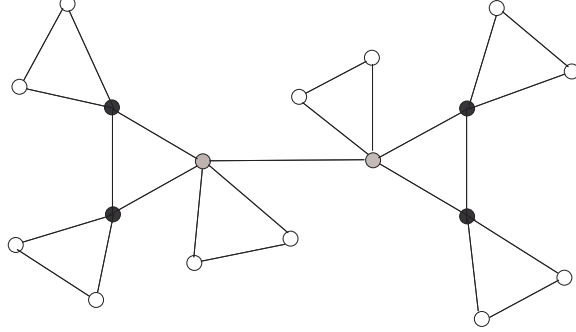


FIGURE 1. $P_2 \odot^2 P_2$

We first recall the useful result obtained in [8].

Proposition 2.1. [8] *Let G be a connected graph of order $n \geq 2$. Then*

- (a) $Z(G) = 1$ if and only if $G = P_n$,
- (b) $Z(G) = n - 1$ if and only if $G = K_n$.

Note that for a connected graph G of order n , we have

$$1 \leq Z(G) \leq n - 1. \quad (1)$$

Lemma 2.2. *Let G be a connected graph of order $n \geq 2$ and let H be a graph of order at least two. Let H_i be the subgraph of $G \odot H$ corresponding to the i^{th} -copy of H .*

- (i) *If S is a zero forcing set of $G \odot H$, then $V_i \cap S \neq \emptyset$ for every $i \in \{1, \dots, n\}$.*
- (ii) *If H is a connected graph and S is a zero forcing set of $G \odot H$, then for every $i \in \{1, \dots, n\}$, $S \cap V_i$ is a zero forcing set of H_i .*

Proof. (i) Suppose $V_i \cap S = \emptyset$, for some i . Then the vertex $v_i \in V$, initially black or forced to black in the zero forcing process, will not force any vertex in V_i to turn black because it has more than one white neighbors, a contradiction. (ii) Suppose contrary that $S \cap V_i$ is not a zero forcing set for H_i . Then there exists a black vertex say $u_i^i \in V_i$, that has more than one white neighbors in graph H_i , so no forcing situation can occur. Note that the vertex $u_i^i \in V_i$ also has more than one white neighbors in $G \odot H$, a contradiction. \square

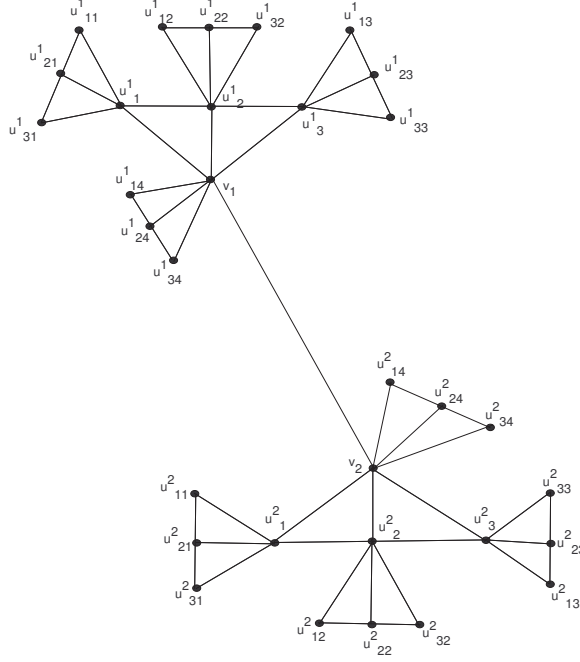


FIGURE 2. Graph $P_2 \odot^2 P_3$

Here we introduce some terminology related to the following theorem. We assume that $B' = \{v_1, v_2, \dots, v_t\}$ is a forcing basis for G and by using B' one can color all the other vertices of G by a sequence of forces in the following order: $v_{t+1}, v_{t+2}, \dots, v_{n_1}$, with appropriate indexing of vertices. We denote the vertices of i -th copies of H in l -th corona ($1 \leq l \leq k$) by $u_{j_1 j_2 \dots j_l}^i$, where $1 \leq j_1 \leq n_2$ and $1 \leq j_p \leq n_2 + 1$ for each $2 \leq p \leq l$. Assume that $Z(H) = m$ and a forcing basis for i^{th} -copy H_i of H in 1st-corona is denoted by $B^i = \{u_1^i, u_2^i, \dots, u_m^i\}$ and forcing basis for i -th copies of H in l^{th} -corona ($2 \leq l \leq k$) by $B_{j_2 \dots j_l}^i = \{u_{1j_2 j_3 \dots j_l}^i, u_{2j_2 j_3 \dots j_l}^i, \dots, u_{mj_2 j_3 \dots j_l}^i\}$, where $1 \leq j_2, \dots, j_l \leq n_2 + 1$. We denote the collection of forcing basis of all copies of H in first corona by B_1 i.e $\cup_{i=1}^{n_1} B^i = B_1$, similarly the collection of forcing basis of all copies of H in l -th corona by B_l i.e $\cup_{i=1}^{n_1} (\cup_{j_l=1}^{n_2+1} \dots (\cup_{j_2=1}^{n_2+1} B_{j_2 \dots j_l}^i)) = B_l$ and $|B_l| = n_1(n_2 + 1)^{l-1}Z(H)$. FIGURE 2 helps in understanding the indices as mentioned.

Theorem 2.3. *Let G and H be connected graphs of order n_1 and n_2 respectively, then*

$$Z(G \odot^k H) = Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1}Z(H).$$

Proof. We prove the result by mathematical induction. For $k = 1$, we have to show that

$$Z(G \odot H) = Z(G) + n_1 Z(H). \quad (2)$$

First, we show that

$$Z(G \odot H) \leq Z(G) + n_1 Z(H). \quad (3)$$

We define $B = B' \cup B_1$ and $|B| = Z(G) + n_1 Z(H)$. We claim that B is a zero forcing set of $G \odot H$. To prove the claim, first assume that B is initially colored black and we color all the vertices of H_i with $1 \leq i \leq t$ which are associated with the vertices of B' using the corresponding sets B^i . Now all the vertices in H_i associated with v_i , $1 \leq i \leq t$ are colored black. Note that there is a vertex v_i belonging to B' that has only one white neighbor v_{t+1} . Thus $v_i \rightarrow v_{t+1}$. Then we color all the vertices in H_{t+1} by using the black vertices in B^{t+1} . Continuing this process, we can color all the vertices of $G \odot H$.

Note that the degree of each vertex of G is increased by n_2 and the degree of each vertex of H is increased by 1 in $G \odot H$. Let $v_i \in B'$ and consider the corresponding copy H_i of H . Note that at least $Z(H) + 1$ vertices are required as initially colored black to start the zero forcing process in each of these H_i 's, $1 \leq i \leq t$. Then $v_t \rightarrow v_{t+1}$ and to continue the process at least $Z(H)$ more vertices are required in H_{t+1} as initially colored black. Continue the process until all the vertices are turned black. Hence,

$$Z(G \odot H) \geq Z(G) + n_1 Z(H). \quad (4)$$

By (3) and (4), (2) holds.

Suppose that the result is true for $k - 1$, i.e

$$Z(G \odot^{k-1} H) = Z(G \odot^{k-2} H) + n_1(n_2 + 1)^{k-2} Z(H). \quad (5)$$

Now we have to show that the result is true for k , i.e

$$Z(G \odot^k H) = Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1} Z(H). \quad (6)$$

We define $B^\varnothing = B' \cup B_1 \cup B_2 \cup \dots \cup B_{(k-1)}$ and $|B^\varnothing| = Z(G) + n_1 Z(H) + n_1(n_2 + 1)Z(H) + \dots + n_1(n_2 + 1)^{k-2} Z(H) = \alpha$. Then B^\varnothing is a zero forcing set of $G \odot^{k-1} H$ by (5). Therefore, we color all the vertices of $G \odot^{k-1} H$ using B^\varnothing . Now all the vertices from 1st-corona to $(k - 1)^{th}$ -corona in $G \odot^k H$ are colored black. It suffices to show that $n_1 n_2 (n_2 + 1)^{k-1}$ vertices in the copies of H in k^{th} -corona in $G \odot^k H$ will be colored black by taking $n_1(n_2 + 1)^{k-1} Z(H)$ more vertices as initially colored black.

First, we show that

$$Z(G \odot^k H) \leq Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1}Z(H). \quad (7)$$

We define $B = B^\varphi \cup B_k$ where $|B| = \alpha + n_1(n_2 + 1)^{k-1}Z(H)$. We claim that B is a zero forcing set of $G \odot^k H$. Let B be initially colored black. Note that the degree of each vertex in H_i in k^{th} -corona in $G \odot^k H$ is increased by one. We color all the vertices of copies of H in k^{th} -corona by using $B_{j_2 \dots j_k}^i$ and the corresponding root vertex in $(k-1)^{th}$ -corona. We obtain the derived set of all black vertices in $G \odot^k H$ resulting from repeatedly applying the color-change rule. Hence, (7) holds.

Note that in each copy of H in k^{th} -corona at least $Z(H)$ vertices are required as initially colored black to continue the zero forcing process. Hence,

$$Z(G \odot^k H) \geq Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1}Z(H). \quad (8)$$

By (7) and (8), (6) holds. Hence, the result is true for any positive integer k . \square

By using Theorem 2.3 and Proposition 2.1, we have the following immediate corollaries:

Corollary 2.4. *Let G and H be connected graphs of order $n_1, n_2 \geq 2$ respectively. Then $Z(G \odot^k H) = Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1}$ if and only if $H \cong P_{n_2}$.*

Corollary 2.5. *Let G and H be connected graphs of order $n_1, n_2 \geq 2$ respectively. Then $Z(G \odot^k H) = Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1}(n_2 - 1)$ if and only if $H \cong K_{n_2}$.*

The wheel graph of order $n + 1$ is defined as $W_{1,n} = K_1 \odot C_n$, where K_1 is the singleton graph. Any three pairwise adjacent vertices of a wheel form a zero forcing set of $W_{1,n}$.

Remark 2.6. *Let $W_{1,n}$, $n \geq 3$, be a wheel graph. Then $Z(W_{1,n}) = 3$.*

The fan graph F_{n_1, n_2} is defined as the join graph $\overline{K_{n_1}} + P_{n_2}$. The case $n_1 = 1$ corresponds to the usual fan graph F_{1, n_2} . Note that $F_{1, n_2} = K_1 \odot P_{n_2}$, where K_1 is the singleton graph. Two adjacent vertices of P_{n_2} where one must be an end vertex form a zero forcing set of F_{1, n_2} .

Remark 2.7. *Let $F_{1,n}$, $n \geq 2$, be a fan graph. Then $Z(F_{1,n}) = 2$.*

Theorem 2.8. *Let G be a connected graph of order $n_1 \geq 2$ and let H be a graph of order $n_2 \geq 2$. Then*

$$Z(G \odot^k H) \leq n_1(n_2 + 1)^{k-1}Z(K_1 \odot H).$$

Proof. We denote by $K_1 \odot H_i$ the subgraph of $G \odot H$, obtained by joining the vertex $v_i \in V$ with all the vertices of H_i . Let A_i be a forcing basis for $K_1 \odot H_i$ and $A = \cup_{i=1}^{n_1} A_i$ with $|A| = n_1 Z(K_1 \odot H)$. We show that A is a zero forcing set of $G \odot H$. Now there are two cases:

Case 1: H contains no isolated vertex. We have that $v_i \in A_i$. So A contains all the vertices of G . Note that any black vertex in H_i has only one white neighbor, so after finite many applications of the color-change rule all the vertices in H_i for each $i = 1, 2, \dots, n_1$ are turned black. Now all the vertices in $G \odot H$ are colored black.

Case 2: H contains isolated vertices, so there exists at least one vertex $x \in H$ such that $x \approx u$, for all $u \in H$. We have that v_i does not belong to the zero forcing set of minimum cardinality of $K_1 \odot H_i$. So A does not contain any vertex from G .

Subcase 2.1: H has only one isolated vertex. Let x_i is the isolated vertex of H_i also $x_i \in A_i$ and $x_i \approx u_p^i$ for any $u_p^i \in V_i$. Thus $x_i \rightarrow v_i$, $1 \leq i \leq n_1$. Now all the vertices of G are forced to black. Note that any black vertex in H_i has only one white neighbor, so after finite many applications of the color-change rule all the vertices in H_i for each $i = 1, 2, \dots, n_1$ are turned black. Now all the vertices in $G \odot H$ are colored black.

Subcase 2.2: H has more than one isolated vertices, then all isolated vertices in H_i belong to A_i except one, say y_i does not belong to A_i , then v_i will be forced by any isolated black vertex in H_i , $1 \leq i \leq n_1$. Now all the vertices of G are forced to black and after finite iterative applications of the color-change rule for connected subgraph of H_i all the vertices in these graphs are turned black for each $i = 1, 2, \dots, n_1$, and then $v_i \rightarrow y_i$. Hence $Z(G \odot H) \leq n_1 Z(K_1 \odot H)$. Therefore, the result follows. \square

Corollary 2.9. *Let G be a connected graph of order $n_1 \geq 2$ and H be a disconnected graph of order $n_2 \geq 2$. Then*

$$Z(G \odot^k H) = n_1(n_2 + 1)^{k-1} Z(K_1 \odot H) = n_1(n_2 + 1)^{k-1}(n_2 - 1)$$

if and only if $H \cong \overline{K_{n_2}}$.

Proof. Suppose $H \cong \overline{K_{n_2}}$. For $k = 1$, we have to show that $Z(G \odot H) = n_1(n_2 - 1)$. We define $B_i = V_i - \{u_l^i\}$, for any $1 \leq l \leq n_2$, and for each $i = 1, 2, \dots, n_1$ and $B = \cup_{i=1}^{n_1} B_i$. We claim that B is a zero forcing set of $G \odot H$ with $|B| = n_1(n_2 - 1)$. To prove the claim, we first assume that B is initially colored black. Note that every initial black vertex of H_i has single white neighbor v_i , so $u_l^i \rightarrow v_i$. Now there is only one white vertex u_l^i in each

V_i and this vertex is the single white neighbor of the corresponding vertex v_i , so $v_i \rightarrow u_l^i$ for each $i = 1, 2, \dots, n_1$.

Consider $v_i \in V$ and the corresponding copy H_i of H for any i , $1 \leq i \leq n_1$. Note that at least $n_2 - 1$ vertices of H_i are required to start the zero forcing process. Hence, $Z(G \odot H) \geq n_1(n_2 - 1)$. Therefore, the result follows.

On the other hand, $Z(G \odot^k H) = n_1(n_2 - 1)$ implies $H \cong \overline{K_{n_2}}$. Suppose $H \not\cong \overline{K_{n_2}}$ and let $u_{(n_2-2)}^i \sim u_{n_2}^i$, for $1 \leq i \leq n_1$. We define $B_i = \{u_1^i, u_2^i, \dots, u_{(n_2-1)}^i\}$, $1 \leq i \leq n_1 - 1$ and $B_{n_1} = \{u_1^{n_1}, u_2^{n_1}, \dots, u_{(n_2-2)}^{n_1}\}$. Let $B = \cup_{i=1}^{n_1-1} B_i \cup B_{n_1}$ with $|B| = n_1(n_2 - 1) - 1$. We show that B is a zero forcing set of $G \odot H$. Assume that B is initially colored black. Note that each isolated vertex in each H_i , $1 \leq i \leq n_1$, has only one white neighbor $v_i \in V$, so v_i will be forced to black for each $i = 1, 2, \dots, n_1$. Now all the vertices of G are colored black. Since $u_{(n_2-2)}^i$ has only one white neighbor $u_{n_2}^i$ for each $i = 1, 2, \dots, n_1$, so $u_{(n_2-2)}^i \rightarrow u_{n_2}^i$, $1 \leq i \leq n_1$. Note that v_{n_1} has only one white neighbor $u_{(n_2-1)}^{n_1}$, so $v_{n_1} \rightarrow u_{(n_2-1)}^{n_1}$. Now all the vertices are colored black. So B is a zero forcing set of cardinality $n_1(n_2 - 1) - 1$, a contradiction. Therefore, the result follows. \square

The following definitions are introduced in [5]. Fix a graph T . A vertex of degree at least three is called a major vertex. An end vertex u is called a terminal vertex of a major vertex v if $d(u, v) < d(u, w)$ for every other major vertex w . The terminal degree of a major vertex v in T , denoted by $ter_T(v)$, is the number of terminal vertices of v . A major vertex v is an exterior major vertex (emv) if it has positive terminal degree. Let $\sigma(G)$ denote the sum of terminal degrees of all major vertices of G and let $ex(G)$ denote the number of emvs of G . We further define an exterior degree two vertex to be a vertex of degree two that lies on a path from a terminal vertex to its major vertex and an interior degree two vertex to be a vertex of degree two such that the shortest path to any terminal vertex includes a major vertex.

Theorem 2.10. [4, 11, 13] *If T is a tree that is not a path, then $\dim(T) = \sigma(T) - ex(T)$.*

Theorem 2.11. [8] *For any tree T , we have $Z(T) = \dim(T)$ iff T has no interior degree two vertices and each major vertex v of T satisfies $ter_T(v) \geq 2$.*

Theorem 2.12. *Let T be a tree of order $n \geq 3$, that has no interior degree two vertices and each major vertex v of T satisfies $ter_T(v) \geq 2$, then*

$$Z(T \odot^k K_1) = \begin{cases} \sigma(T), & k = 1, \\ 2^{k-2}n, & k \geq 2. \end{cases}$$

Proof. Since $T \odot^k K_1$ is a tree, with no interior degree two vertex and each major vertex v satisfies $\text{ter}_T(v) \geq 2$. Now for $k = 1$, $\sigma(T \odot K_1) = n$ and $\text{ex}(T \odot K_1) = n - \sigma(T)$, $\dim(T \odot K_1) = \sigma(T) = Z(T \odot K_1)$ by Theorem 2.10 and Theorem 2.11 we obtain the result. Since we have $\sigma(T \odot^2 K_1) = 2n$, $\text{ex}(T \odot^2 K_1) = n$, so we obtain the result for $k = 2$ by Theorem 2.10 and Theorem 2.11, $\dim(T \odot^2 K_1) = n = Z(T \odot^2 K_1)$. \square

Let α be the number of connected components of a graph H . Let us denote the connected components of H by C_l , where $1 \leq l \leq \alpha$.

Theorem 2.13. *Let G be a connected graph of order n_1 and H be a graph of order n_2 . Let α be the number of connected components of H of order greater than one and let β be the number of isolated vertices of H . Then*

$$Z(G \odot^k H) \leq \begin{cases} n_1(n_2 + 1)^{k-1} \sum_{l=1}^{\alpha} Z(C_l) + n_1(n_2 + 1)^{k-1}(\beta - 1), & \alpha \geq 1, \beta \geq 2, \\ Z(G \odot^{k-1} H) + n_1(n_2 + 1)^{k-1} \sum_{l=1}^{\alpha} Z(C_l), & \alpha \geq 1, \beta = 0, \\ n_1(n_2 + 1)^{k-1} \sum_{l=1}^{\alpha} Z(C_l) + n_1(n_2 + 1)^{k-1} - 1, & \alpha \geq 1, \beta = 1, \\ n_1(n_2 + 1)^{k-1}(n_2 - 1), & \alpha = 0, \beta \geq 2. \end{cases}$$

Proof. We define K_l^i , $1 \leq l \leq \alpha$, be a forcing basis for connected component C_l^i of H_i , $1 \leq i \leq n_1$.

We suppose $\alpha \geq 1, \beta \geq 2$. We define P_i to be the set of vertices of $G \odot H$ formed by all but one of the isolated vertices of H_i , $1 \leq i \leq n_1$. Let us show that $B = \cup_{i=1}^{n_1} (\cup_{l=1}^{\alpha} K_l^i \cup P_i)$ is a zero forcing set of $G \odot H$ with $|B| = n_1 \sum_{l=1}^{\alpha} Z(C_l) + n_1(\beta - 1)$. Let B is initially colored black. Note that $p_1^i \in P_i$ has only one white neighbor $v_i \in V$, so $p_1^i \rightarrow v_i$ for each i , $1 \leq i \leq n_1$. Now all the vertices of G are colored black. We color all the vertices of connected components C_l^i , $1 \leq l \leq \alpha$, of H_i using K_l^i and the corresponding vertex $v_i \in V$, $1 \leq i \leq n_1$. Note that the vertex v_i has only one white neighbor p_{β}^i , the isolated vertex of H_i not belonging to P_i , so $v_i \rightarrow p_{\beta}^i$, $1 \leq i \leq n_1$, and we have the derived set of all black vertices in $G \odot H$. As a consequence, $Z(G \odot H) \leq n_1 \sum_{l=1}^{\alpha} Z(C_l) + n_1(\beta - 1)$. Therefore, the result follows.

Now suppose $\alpha \geq 1$, and $\beta = 0$. Let $B = B' \cup_{i=1}^{n_1} (\cup_{l=1}^{\alpha} K_l^i)$, where $B' = \{v_1, v_2, \dots, v_t\}$ is a forcing basis for G and by using B' one can color all the other vertices of G by a sequence of forces in the following order: $v_{t+1}, v_{t+2}, \dots, v_n$, with appropriate indexing of vertices. We show that B is a zero forcing set of $G \odot H$. Consider iterative applications of the color-change

rule with initial black set B . We color all the vertices of H_i with $1 \leq i \leq t$ which are associated with the vertices of B' using the corresponding sets K_l^i , $1 \leq l \leq \alpha$. Now all the vertices in H_i associated with v_i , $1 \leq i \leq t$ are colored black. Note that there is a vertex v_i belonging to B' that has only one white neighbor $v_{t+1} \in V$. Thus $v_i \rightarrow v_{t+1}$. Then we color all the vertices in H_{t+1} using the black vertices in K_l^{t+1} , $1 \leq l \leq \alpha$. Continuing this process, we can color all the vertices of $G \odot H$. So $Z(G \odot H) \leq Z(G) + n_1 \sum_{l=1}^{\alpha} Z(C_l)$. Therefore, the result follows.

Now suppose $\alpha \geq 1$, $\beta = 1$. Let R be the set of all isolated vertices in each H_i , $1 \leq i \leq n_1$, except one say r_{n_1} . We define $B = \cup_{i=1}^{n_1} (\cup_{l=1}^{\alpha} K_l^i) \cup R$. We show that B is a zero forcing set of $G \odot H$. Let B is initially colored black. Note that the vertex $r_i \in R$, $1 \leq i \leq n_1 - 1$, has only one white neighbor $v_i \in V$, so $r_i \rightarrow v_i$, $1 \leq i \leq n_1 - 1$. We color all the vertices of H_i , $1 \leq i \leq n_1 - 1$, which are associated with v_i , $1 \leq i \leq n_1 - 1$ using the corresponding sets K_l^i , $1 \leq l \leq \alpha$. Now all the vertices in H_i associated with v_i , $1 \leq i \leq n_1 - 1$ are colored black. Now v_{n_1-1} has only one white neighbor v_{n_1} , so $v_{n_1-1} \rightarrow v_{n_1}$. We color $C_l^{n_1}$ in H_{n_1} using $K_l^{n_1}$, $1 \leq l \leq \alpha$. Now v_{n_1} has only one white neighbor $r_{n_1} \in V_{n_1}$, so $v_{n_1} \rightarrow r_{n_1}$ and we have the derived set of all black vertices. So $Z(G \odot H) \leq n_1 \sum_{l=1}^{\alpha} Z(C_l) + n_1 - 1$. Therefore, the result follows.

Now suppose $\alpha = 0$, $\beta \geq 2$. Here H is an empty graph so the result followed by Corollary 2.9. \square

3. LEXICOGRAPHIC PRODUCT OF GRAPHS

Let G and H be two graphs. The lexicographic product of G and H , denoted by $G \circ H$, is the graph with vertex set $V(G) \times V(H) = \{(a, v) \mid a \in V(G) \text{ and } v \in V(H)\}$, where (a, v) is adjacent to (b, w) whenever $ab \in E(G)$ or $a = b$ and $vw \in E(H)$. For any vertex $a \in V(G)$ and $b \in V(H)$, we define the vertex set $H(a) = \{(a, v) \in V(G \circ H) \mid v \in V(H)\}$ and $G(b) = \{(v, b) \in V(G \circ H) \mid v \in V(G)\}$. It is clear that the graph induced by $H(a)$, called a layer $H(a)$, is isomorphic to H and the graph induced by $G(b)$, called a layer $G(b)$, is isomorphic to G , denoted by $H(a) \cong H$ and $G(b) \cong G$ respectively. We write $H(a) \sim H(b)$ when each vertex of $H(a)$ is adjacent to all vertices of $H(b)$ and vice versa, and $H(a) \not\sim H(b)$ means that no vertex of $H(a)$ is adjacent to any vertex of $H(b)$ and vice versa.

Let G be a connected graph and H be a non-trivial graph containing $k \geq 1$ components H_1, H_2, \dots, H_k with $|V(H_j)| \geq 2$ for each $j = 1, 2, \dots, k$. For any vertex $a \in V(G)$ and $1 \leq i \leq k$, we define the vertex set $H_i(a) = \{(a, v) \in$

$V(G \circ H) \setminus \{v \in V(H_i)\}$. Let $|V(H_i)| = m_i$, $1 \leq i \leq k$. From the definition of $G \circ H$, it is clear that for every $(a, v) \in V(G \circ H)$, $\deg_{G \circ H}(a, v) = \deg_G(a) \cdot |V(H)| + \deg_H(v)$. If G is a disconnected graph having $k \geq 2$ components G_1, G_2, \dots, G_k , then $G \circ H$ is also a disconnected graph having k components such that $G \circ H = G_1 \circ H \cup G_2 \circ H \cup \dots \cup G_k \circ H$ and each component $G_i \circ H$ is the lexicographic product of connected component G_i of G with H , therefore throughout this section, we will assume G to be connected.

Observation 3.1. *For any $a, b \in V(G)$, either $H(a) \sim H(b)$ or $H(a) \not\sim H(b)$ in $G \circ H$.*

First, we give a general lower bound on the zero forcing number of lexicographic product of graphs. Note that given any connected graph G , then $Z(G) = 1$ if and only if $G \cong P_n$, $n \geq 2$. So, if $Z(G \circ H) = 1$ for some graph H , then clearly $G \circ H$ is a path graph, i.e G is the trivial graph K_1 and H is a path or viceversa. So, we have the following result:

Remark 3.2. *If G and H are non trivial graphs, then $Z(G \circ H) \geq 2$.*

Lemma 3.3. *Let G be a connected graph on n vertices. There exists a forcing basis S for $G + K_1$ such that $S \subseteq V(G)$.*

Proof. Let $V(G + K_1) = V(G) \cup \{v\}$. If $v \notin S$ we have nothing to prove. Suppose that $v \in S$. Since G is connected and $\deg_{G+K_1}(v) = n$ and also v is initially colored black so by equation (1), there exists at least one white vertex $x \in N_{G+K_1}(v)$ such that $(S \setminus \{v\}) \cup \{x\}$ is a forcing basis for $G + K_1$. \square

Theorem 3.4. *Let G be a connected graph and H be an arbitrary graph containing $k \geq 1$ components $H_1, H_2, H_3, \dots, H_k$ and $m_i \geq 2$. Let Z be a zero forcing set of $G \circ H$. For any vertex $a \in V(G)$, if $Z_i(a) = Z \cap H_i(a)$ for every $i \in \{1, 2, \dots, k\}$, then $Z_i(a) \neq \emptyset$. Moreover, if B_i is a forcing basis for H_i , then $|Z_i(a)| \geq |B_i|$.*

Proof. Suppose that for some $i \in \{1, 2, \dots, k\}$ there exists a vertex $a \in V(G)$ such that $Z_i(a) = \emptyset$. Then, by Observation 3.1 any vertex in $H_i(a)$ cannot be forced by any vertex in $H_j(b)$, $i \neq j$ for any $a \neq b \in V(G)$, a contradiction.

Now suppose that $|Z_i(a)| < |B_i|$ and $Z_i(a) = \{(a, z_1), (a, z_2), \dots, (a, z_t)\}$ for some forcing basis B_i of H_i , where $\{z_1, z_2, \dots, z_t\} \subset V(H_i)$. Then, each black vertex in $H_i(a)$ has more than one white neighbors and no vertex of $H_i(a)$ can be forced by any vertex in $H_j(v)$ for any $v \in V(G)$, $i \neq j$. Hence, $|Z_i(a)| \geq |B_i|$. \square

From above theorem, we have an immediate corollary:

Corollary 3.5. *Let G be a connected graph and H be an arbitrary graph containing $k \geq 1$ components $H_1, H_2, H_3, \dots, H_k$ and $m_i \geq 2$. Let $Z(a) = \bigcup_{1 \leq i \leq k} Z_i(a)$ for $a \in V(G)$. Then $Z(a)$ is a zero forcing set of $H(a)$.*

Proposition 3.6. *Let G be a connected graph and H be an arbitrary graph containing $k \geq 1$ components H_1, H_2, \dots, H_k and $m_i \geq 2$. Let $a \in V(G)$ and Z be a forcing basis for $G \circ H$. If $Z(a) = Z \cap H(a)$ and $\alpha(a) = |Z(a)|$. Then*

$$\alpha(a) \leq \sum_{i=1}^k m_i.$$

Proof. For any $(b, x) \in V(G \circ H)$, $\deg_{G \circ H}(b, x) = \sum_{b \sim u} |H(u)| + \deg_H(x)$. Since G is connected so for any $b \in V(G)$, there exist at least one vertex $v \in V(G)$ such that $b \sim v$ and $\deg_{G \circ H}(b, x) \geq |H(v)| + \deg_H(x)$. To start the zero forcing process at least all the vertices of $H(v)$ along with $\deg_H(x)$ vertices are initially colored black. Since Z is a forcing basis for $G \circ H$. Hence, $Z \cap H(v) = H(v)$ and $\alpha(a) \leq \sum_{i=1}^k m_i$ for any $a \in V(G)$. \square

Corollary 3.7. *Let G be a connected graph and H be an arbitrary graph containing $k \geq 1$ components H_1, H_2, \dots, H_k and $m_i \geq 2$. Then there exists at least one vertex $x \in V(G)$ such that $\alpha(x) = \sum_{i=1}^k m_i$.*

The projection of $S \subseteq V(G \circ H)$ onto G , denoted by $P_G(S)$, is the set of vertices $a \in V(G)$ for which there exists a vertex $(a, v) \in S$. Similarly, the projection of $S \subseteq V(G \circ H)$ onto H , $P_H(S)$, is the set of vertices $v \in V(H)$ for which there exists a vertex $(a, v) \in S$.

Lemma 3.8. *Let G be a connected graph of order n and H be an arbitrary graph containing $k \geq 1$ components H_1, H_2, \dots, H_k and $m_i \geq 2$. Let Z be a forcing basis for $G \circ H$ and $Z_i = Z \cap V(G \circ H_i)$, where $G \circ H_i$ is the induced subgraph of $G \circ H$. Then $P_G(Z_i) = V(G)$.*

Proof. Let $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(H_i) = \{v_1^i, v_2^i, \dots, v_{m_i}^i\}$ for $1 \leq i \leq k$. Suppose $P_G(Z_i) \neq V(G)$, i.e there exists a vertex $u_j \in V(G)$ such that $u_j \notin P_G(Z_i)$. This implies $(u_j, v_p^i) \notin Z$ for any $v_p^i \in V(H_i)$ for $1 \leq p \leq m_i$. Hence, $H_i(u_j) \cap Z = \phi$, a contradiction by Theorem 3.4. \square

Lemma 3.9. *Let G be a connected graph of order n and H be an arbitrary graph containing $k \geq 1$ components H_1, H_2, \dots, H_k and $m_i \geq 2$. Then*

$$Z(G \circ H) \leq n \left(\sum_{i=1}^k m_i \right) - k.$$

Proof. Let $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(H_i) = \{v_1^i, v_2^i, \dots, v_{m_i}^i\}$ for $1 \leq i \leq k$. We define $Z = V(G \circ H) \setminus \{(u_1, v_2^i) | 1 \leq i \leq k\}$ and $|Z| = n(\sum_{i=1}^k m_i) - k$. We claim that Z is a zero forcing set of $G \circ H$. To prove the claim, assume that Z is initially colored black. Since for any $i \neq j$, $v_p^i \not\sim v_q^j$ in H hence $(u_r, v_p^i) \not\sim (u_r, v_q^j)$, $1 \leq r \leq n$, in $G \circ H$. Since for any i , H_i is connected, so there exists at least one vertex v_l^i such that $v_2^i \sim v_l^i$ in H_i and hence $(u_r, v_2^i) \sim (u_r, v_l^i)$, $1 \leq r \leq n$, in $G \circ H$. Therefore, $(u_1, v_l^i) \rightarrow (u_1, v_2^i)$ for $1 \leq i \leq k$. Hence, $Z(G \circ H) \leq n(\sum_{i=1}^k m_i) - k$. \square

This bound is sharp for $G = K_n$ and $H = K_{m_1}, K_{m_2}, \dots, K_{m_k}$.

Lemma 3.10. *Let G be a connected graph of order n and H be an arbitrary graph containing $k \geq 1$ components H_1, H_2, \dots, H_k and $m_i \geq 2$. Then*

$$Z(G \circ H) \geq (n-1)k + \sum_{i=1}^k m_i.$$

Proof. The result follows from Corollary 3.7 and Lemma 3.8. \square

This bound is sharp for $G = K_{1,n-1}$ and $H = P_{m_1}, P_{m_2}, \dots, P_{m_k}$.

Lemma 3.11. *Let G be a connected graph of order n and H_1, H_2, \dots, H_k , $k \geq 2$ are singleton components. Then $Z(G \circ H) \leq nk - 2$.*

Proof. Let $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(H_i) = \{x_i\}$ for $1 \leq i \leq k$. We define, for $u_1 \sim u_2$, $Z = V(G \circ H) \setminus \{(u_1, x_k), (u_2, x_k)\}$ with $|Z| = nk - 2$. Since $x_1 \not\sim x_k$ so $(u_1, x_1) \rightarrow (u_2, x_k)$. Similarly, $(u_2, x_1) \rightarrow (u_1, x_k)$. Now all the vertices in $G \circ H$ are colored black. Therefore, the result follows. \square

Now we study the zero forcing number of lexicographic product of graphs for some specific families of graphs and H contains one component only. Note that $K_m \circ K_n \cong K_{mn}$, so $Z(K_M \circ K_n) = mn - 1$. Therefore, from now on we consider the graphs when at most one of the factors of the product is a complete graph.

Lemma 3.12. *For any connected graph H of order m , $Z(K_n \circ H) = Z(H) + (n-1)m$.*

Proof. Note that for any $a \in V(K_n)$, $H(a) \sim H(b)$ for all $b \in V(K_n) \setminus \{a\}$. Therefore all the vertices in $H(b)$ for all $b \in V(K_n) \setminus \{a\}$ are initially colored black. Now $H(a) \cong H$ so $Z(H)$ vertices are required as initially colored black to complete the zero forcing process. \square

Since the lexicographic product of graphs is not commutative, i.e $K_n \circ H \not\cong H \circ K_n$. Therefore we study the case when the second factor is a complete graph.

Lemma 3.13. *For any connected non complete graph G of order m , $m(n-1)+1 \leq Z(G \circ K_n) \leq nm-2$.*

Proof. Suppose $V(G) = \{u_1, u_2, \dots, u_m\}$ and $V(K_n) = \{v_1, v_2, \dots, v_n\}$. It is easy to check that $Z = V(G \circ K_n) \setminus \{(u_1, v_2), (u_n, v_2)\}$ is a zero forcing set of $G \circ K_n$. Hence $Z(G \circ K_n) \leq nm-2$.

By Corollary 3.7, there exists at least one vertex $a \in V(G)$ such that $\alpha(a) = n$. Since $H(v) \cong K_n$, therefore for $m-1$ layers $H(v)$ at least $n-1$ vertices from each layer are required as initially colored black to color all the vertices of $G \circ K_n$. Hence $Z(G \circ K_n) \geq n + (m-1)(n-1) = m(n-1) + 1$. \square

Now we study the zero forcing number of $P_n \circ H$, for $n \geq 3$ and a connected graph H . Suppose $V(P_n) = \{u_1, u_2, \dots, u_n\}$. Since $H(u_i) \sim H(u_{i+1})$, $1 \leq i \leq n-1$. Note that to color the vertices of $H(u_1)$, $Z(H)$ vertices in $H(u_1)$ and all the vertices in $H(u_2)$ are required as initially colored black and to color the vertices of $H(u_3)$, $Z(H)$ vertices in $H(u_3)$ and all the vertices in $H(u_4)$ are required as initially colored black. Continue the process untill all the vertices in $P_n \circ H$ are turned black.

Proposition 3.14. *For a connected graph H and $n \geq 3$,*

$$Z(P_n \circ H) = \begin{cases} \frac{n(Z(H)+m)}{2}, & n \text{ is even} \\ \frac{n(Z(H)+m)+Z(H)-m}{2}, & n \text{ is odd.} \end{cases}$$

Corollary 3.15. *For $n, m \geq 3$,*

$$Z(P_n \circ K_m) = \begin{cases} \frac{n(2m-1)}{2}, & n \text{ is even} \\ nm - \frac{n+1}{2}, & n \text{ is odd.} \end{cases}$$

Now we study the zero forcing number of $C_n \circ H$, for $n \geq 4$ and a connected graph H . Suppose $V(C_n) = \{u_1, u_2, \dots, u_n\}$. Since $H(u_i) \sim H(u_{i+1})$, $1 \leq i \leq n$ and $u_{n+1} = u_1$. Note that to color the vertices of $H(u_1)$, $Z(H)$ vertices in $H(u_1)$ and all the vertices in $H(u_2)$ and $H(u_n)$ are required as initially colored black and to color the vertices of $H(u_3)$, $Z(H)$ vertices in $H(u_3)$ and all the vertices in $H(u_4)$ are required as initially colored black. Continue the process untill all the vertices in $C_n \circ H$ are turned black.

Proposition 3.16. *For a connected graph H and $n \geq 4$,*

$$Z(C_n \circ H) = \begin{cases} \frac{n(Z(H)+m)}{2}, & n \text{ is even} \\ \frac{n(m+Z(H))+m-Z(H)}{2}, & n \text{ is odd.} \end{cases}$$

Corollary 3.17. *For $n \geq 4$ and $m \geq 3$,*

$$Z(C_n \circ K_m) = \begin{cases} \frac{n(2m-1)}{2}, & n \text{ is even} \\ \frac{n(2m-1)+1}{2}, & n \text{ is odd.} \end{cases}$$

REFERENCES

- [1] AIM Minimum Rank - Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelsen, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen and A. W. Wehe). Zero forcing sets and the minimum rank of graphs, *Linear Algebra Appl.*, 428/7 (2008) 1628 – 1648.
- [2] F. Barioli, W. Barrett, S. M. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche and H. van der Holst, Zero forcing parameters and minimum rank problems, *Linear Algebra Appl.*, 433 (2010) 401 – 411.
- [3] A. Berman, S. Friedland, L. Hogben, U. G. Rothblum and B. Shader, An upper bound for the minimum rank of a graph, *Linear Algebra Appl.*, 429 (2008) 1629 – 1638.
- [4] G. Chartrand, L. Eroh, M. A. Johnson and O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.*, 105 (2000) 99 – 113.
- [5] K. Chilakamarri, N. Dean, C. X. Kang and E. Yi, Iteration index of a zero forcing set in a graph, *Bull. Inst. Combin. Appl.*, 64 (2012) 57 – 72.
- [6] C. J. Edholm, L. Hogben, M. Hyunh, J. LaGrange and D. D. Row, Vertex and edge spread of zero forcing number, maximum nullity, and minimum rank of a graph, *Linear Algebra Appl.*, 436 (2012) 4352 – 4372.
- [7] L. Eroh, C. X. Kang and E. Yi, On zero forcing number of graphs and their complements, *Discrete Math. Algorithm. Appl.*, 07 (2015) DOI: 10.1142/S1793830915500020.
- [8] L. Eroh, C. X. Kang and E. Yi, A comparison between the metric dimension and zero forcing number of trees and unicyclic graphs, arxiv:1408.5943v1 [math.Co].
- [9] S. M. Fallat and L. Hogben, The minimum rank of symmetric matrices described by a graph: A survey, *Linear Algebra Appl.*, 426 (2007) 558 – 582.
- [10] S. M. Fallet and L. Hogben, Variants on the minimum rank problem: A survey II, arXiv:1102.5142v1 [math.Co].
- [11] F. Harary and R. A. Melter, On the metric dimension of a graph, *Ars Combinatoria.*, 2(1976)191 – 195.
- [12] L. Hogben, M. Huynh, N. Kingsley, S. Meyer, S. Walker and M. Young, Propagation time for zero forcing on a graph, *Discrete Appl. Math.*, 160 (2012) 1994 – 2005.
- [13] P. J. Slater, Leaves of trees, *Congr. Numer.*, 14 (1975) 549 – 559.
- [14] F. A. Taklimi, Zero forcing sets for graphs, Ph.D Thesis, University of Regina, August (2013).

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